Almost Shift Invariant Projections in Infinite Tensor Products

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The Rohlin property

Let σ be an automorphism of a unital C^*-algebra A. Following [Con], [HO], [BEK] we will say that the dynamical system (A, σ) has the *Rohlin property* if there exists a strictly increasing sequence $(n_k)_k$ of positive integers such that for each $m = n_k$ there is a sequence

$$(e_i^{(j)} | i = 0, 1, \dots, m)_j$$

of families of projections in A with

$$e_m^{(j)} = e_0^{(j)},$$
$$\sum_{i=0}^{m-1} e_i^{(j)} = 1,$$
$$\lim_{j \to \infty} \|\sigma(e_i^{(j)}) - e_{i+1}^{(j)}\| = 0$$

for $i = 0, 1, \dots, m-1$, and

$$\lim_{j \to \infty} \| [x, e_i^{(j)}] \| = 0$$

for all $x \in A$, $i = 0, 1, \dots, m-1$.

It was established in [Voi, Lemma 3.3] (see also [BKRS], Lemma 2.1]) that if A is a UHF algebra (or more generally, if A is a unital AF algebra and σ is an approximately inner automorphism) and (A, σ) has the Rohlin property, then σ is an almost inductive limit automorphism. This means that for any $B \in \mathcal{F}(A)$ = the set of finite dimensional *-subalgebras of A containing the unit of A, and any $\varepsilon > 0$, there is a $C \in \mathcal{F}(A)$ such that $B \subseteq^{\varepsilon} C$ and $d(\alpha(C), C) < \varepsilon$. Here we use the terminilogy $C_1 \subseteq^{\varepsilon} C_2$ iff

$$\sup\{\inf\{\|x-y\| \big| y \in C_2, \|y\| \leq 1\} \big| x \in C_1, \|x\| \leq 1\} < \varepsilon$$

and

$$d(C_1, C_2) = \inf\{\varepsilon > 0 | C_1 \subseteq^{\varepsilon} C_2 \text{ and } C_2 \subseteq^{\varepsilon} C_1\}$$

whenever C_1, C_2 are subspaces of A. If σ is an almost inductive limit automorphism and $\varepsilon > 0$, then there exists a unitary $v \in A$ such that $Adv \circ \sigma$ is an inductive limit automorphism and $\|v - 1\| < \varepsilon$, [Voi, Proposition 2.3]. That $\beta = Adv \circ \sigma$ is an inductive limit automorphism means that there is an increasing sequence $(A_n) \subseteq \mathcal{F}(A)$ such that $\bigcup_n A_n$ is dense in A and $\beta(A_n) = A_n$ for all n.

It is an open problem whether or not any automorphism of a UHF algebra is an almost inductive limit automorphism. If in particular $A = M_{2^\infty} = \bigotimes^{\infty} M_2$, i.e. A is the CAR algebra, and σ is the (Bernoulli) shift on A, it was established in [BKRS] that σ has the Rohlin property, and hence σ is an almost inductive limit automorphism. Let σ be a quasifree automorphism of A, i.e.

$$\sigma(a(f)) = a(Uf)$$

where $f \in \mathcal{H}$ = the one particle space, $a(f)$ is the annihilator corresponding to f, and U is a unitary operator on $\mathcal{H}$. It was established in [BEK] that σ has the Rohlin property if and only if $U^n - 1$ is not of Hilbert-Schmidt class for all $n \in \mathbf{Z} \setminus \{0\}$, i.e. if and only if σ^n is outer in the trace representation for all $n \in \mathbf{Z} \setminus \{0\}$. Thus σ is an almost inductive limit automorphism in this case. If on the other hand $U^n - 1$ is a Hilbert-Schmidt operator for some $n \neq 0$, then U has purely discrete spectrum and σ is a product type automorphism, and hence σ is a (strict) inductive limit automorphism.

The present paper came out of an attempt to establish that the shift σ on $A = M_{n^\infty} = \bigotimes^{\infty} M_n$ has the Rohlin property also for $n = 3, 5, \ldots$ (when n is even, the Rohlin property follows from the Rohlin property of the shift on M_{2^∞}). We have, however, only been able to establish the following approximate version of the Rohlin property.

Theorem 1. *Let σ be the one-sided shift on M_{2^∞}, and let τ be the trace state on M_{n^∞}. For any finite subset $\{x_1, \ldots, x_k\} \subseteq M_{n^\infty}$, any $m \in \mathbf{N}$ and any $\varepsilon > 0$ there exists projections $e_0, e_1, \ldots, e_{2^m}$ in M_{n^∞} such that*

$$e_{2^m} = e_0\,,$$

$$e = \sum_{i=0}^{2^m-1} e_i \quad \text{is a projection,}$$

$$\tau(e) > 1 - \varepsilon,$$
$$\|\sigma(e_i) - e_{i+1}\| < \varepsilon$$

for $i = 0, 1, \ldots, 2^m - 1$, *and*

$$\|[x_j, e_i]\| < \varepsilon$$

for $i = 0, 1, \ldots, 2^m - 1$, $j = 1, \ldots, k$.

Combining Theorem 1 with the techniques of [Voi], [BKRS] we then establish that the shift is approximately an almost inductive limit automorphism in the following sense:

Corollary 2. *Let σ be the one-sided shift on M_{n^∞}. For any finite dimensional sub-*algebra D of M_{n^∞} and any $\varepsilon > 0$, there exists a projection $e \in D' \cap M_{n^\infty}$ such that*

$$\|\sigma(e) - e\| < \varepsilon$$

and

$$\tau(e) > 1 - \varepsilon,$$

*and there exists a finite-dimensional *-subalgebra $E \subseteq eM_{n^\infty}e$ such that*

$$De \subseteq^{\varepsilon} E$$

and

$$d(E, \sigma(E)) < \varepsilon.$$

The embedding of $GICAR(\mathcal{H})$ into M_{n^∞}

In this section we prove Theorem 1 and Corollary 2. Let σ be the one-sided shift on M_{n^∞}, $n = 2, 3, 4, \ldots$. Let $GICAR(\mathcal{H})$ be the gauge-invariant part of the CAR algebra $CAR(\mathcal{H})$ over the Hilbert space $\mathcal{H}$, i.e., $GICAR(\mathcal{H})$ consists of the elements in $CAR(\mathcal{H})$ which are invariant under the quasi-free action of the circle group T given by

$$a(f) \to a(zf)$$

for $z \in \mathbf{T} \subseteq \mathbf{C}$, $f \in \mathcal{H}$. Following [CE] (see also [E]), $GICAR(\mathcal{H})$ can be embedded into M_{n^∞} as follows: Let $(f_{ij})_{i,j=1}^n$ be a complete set of matrix units for M_n, and define projections $e_1, e_2, \ldots$ in $M_{n^\infty} = \bigotimes_1^\infty M_n$ as follows

$$\begin{aligned} e_1 &= \sum_{ij} \frac{1}{n} f_{ij} \otimes 1 \otimes 1 \otimes \cdots, \\ e_2 &= \sum_{i} f_{ii} \otimes f_{ii} \otimes 1 \otimes \cdots, \end{aligned}$$

$$
\begin{aligned}
e_3 &= \sum_{ij} \frac{1}{n} 1 \otimes f_{ij} \otimes 1 \otimes \cdots , \\
e_4 &= \sum_{i} 1 \otimes f_{ii} \otimes f_{ii} \otimes \cdots ,
\end{aligned}
$$

etc. The e_i's satisfy the Temperley-Lieb relations

$$
\begin{aligned}
&e_i e_j = e_j e_i \qquad \text{if} \quad |i-j| \geq 2 , \\
&e_k e_{k\pm 1} e_k = \frac{1}{n} e_k .
\end{aligned}
$$

Furthermore,

$$\sigma(e_k) = e_{k+2}$$

for $k = 1, 2, \ldots$

Let $f_1, f_2, \ldots$ be a basis for $\mathcal{H}$. The one-sided quasi-free shift is the injective morphism of $CAR(\mathcal{H})$ defined by

$$\beta(a_k) = a_{k+1}$$

where $a_k = a(f_k)$, $k = 1, 2, \ldots$. The shift β restricts to an injective morphism of $GICAR(\mathcal{H})$ which we also denote by β.

The embedding γ of $GICAR(\mathcal{H})$ into M_{n^∞} is given by

$$
\begin{aligned}
&\gamma(a_i^* a_i) = e_{2i} \\
&\gamma(a_i^* a_{i+1}) = e_{2i}(1 - n e_{2i+1}) e_{2i+2} ,
\end{aligned}
$$

[CE]. We outline the argument from [CE]: First note that if $gl(\mathrm{N})$ denotes the Lie algebra of complex $\infty \times \infty$ matrices such that only finitely many matrix elements are non-zero, then

$$H = [H_{ij}] \in gl(\mathrm{N}) \to Q(H) = \sum_{ij} H_{ij} a_i^* a_j$$

is a Lie algebra morphism, i.e.

$$Q([H, H']) = [Q(H), Q(H')] .$$

Since

$$[e_{i,i+1}, e_{i+1,i+2}] = e_{i,i+2}$$

etc., it follows that

$$[a_i^* a_{i+1}, a_{i+1}^* a_{i+2}] = a_i^* a_{i+2}$$

etc. It follows that the *-algebra generated by $a_i^* a_i, a_i^* a_{i+1}$ contains all elements of the form $a_i^* a_j$, $i, j = 1, 2, \ldots$ Since any gauge invariant polynomial in a_i, a_j^*, $i, j = 1, \ldots$ may be written as a polynomial in $a_i^* a_j$ it follows that

We are now ready to construct the approximate Rohlin towers of Theorem 2. For economy of notation we identify $GICAR(\mathcal{H})$ with its image in M_{n_∞}.

If $\lambda, \mu \in \mathbf{T} \subseteq \mathbf{C}$, we may choose unit vectors $f, g \in \mathcal{H}$ such that f, g are orthogonal, and

$$Uf \approx \lambda f\,, \qquad Ug \approx \mu g$$

where U is the isometry implementing the one-sided shift on $\mathcal{H}$. Then

$$v = a^*(f)a(g)$$

is a partial isometry with

$$\sigma(v) \approx \lambda\bar{\mu}v\,.$$

Furthermore

$$\begin{aligned} v^*v &= a(g)^*a(f)a(f)^*a(g) \\ &= a(f)a(f)^*a(g)^*a(g)\,. \end{aligned}$$

Here $a(f)a(f)^*$ and $a(g)^*a(g)$ are commuting projections in different tensor factors (if f, g are chosen suitably) of trace $\frac{n-1}{n}$ and $\frac{1}{n}$, respectively, so

$$\tau(v^*v) = \frac{n-1}{n^2}$$

Also, as

$$\begin{aligned} v^*v &= a(f)^*a(g)a(g)^*a(f) \\ &= a(f)^*a(f)a(g)a(g)^*\,. \end{aligned}$$

and the projections $a(f)^*a(f)$, $a(g)a(g)^*$ are orthogonal to $a(f)a(f)^*$, $a(g)^*a(g)$, respectively, the projection $v\,v^*$ is orthogonal to v^*v,

$$v^*vvv^* = 0 = vv^*v^*v$$

Hence the C^*-algebra $C^*(v)$ generated by v is isomorphic to M_2, and if $1_v = v^*v + vv^*$ is the identity of this C^*-algebra, then

$$\tau(1_v) = \frac{2(n-1)}{n^2}$$

Also

$$\sigma\Big|_{C^*(v)} \approx \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & \lambda\bar{\mu} \end{pmatrix}$$

Now, repeat the construction with $f_2, g_2 \perp f, g$ and (f_2, g_2) orthonormal, and cut down by $1 - 1_v = 1 - vv^* - v^*v$ to obtain a partial isometry v_2 with

$$\tau(v_2v_2^*) = \tau(v_2^*v_2) = \frac{n-1}{n^2}\Big(1 - 2\frac{(n-1)}{n^2}\Big)$$

and

$$\sigma(v_2) \approx \lambda\bar{\mu}v_2$$

so

$$\tau(1 - vv^* - v^*v - v_2v_2^* - v_2^*v_2) = \Big(1 - 2\frac{(n-1)}{n^2}\Big)^2$$

Continuing in this manner, we may construct partial isometries $v_1 = v, \dots, v_m$ such that the projections $v_1v_1^*, v_1^*v_1, v_2v_2^*, v_2^*v_2, \dots, v_mv_m^*, v_m^*v_m$ are all orthogonal, and

$$\sigma(v_k) \approx \lambda\bar{\mu}v_k$$

for all $k = 1, \dots, m$ and

$$\tau\Big(1 - \sum_{k=1}^{m}(v_kv_k^* + v_k^*v_k)\Big) = \Big(1 - 2\frac{n-1}{n^2}\Big)^m$$

Now, put

$$u_m = v_1 + v_2 + \dots + v_m$$

then u_m is a partial isometry such that $u_mu_m^* \perp u_m^*u_m, \sigma(u_m) \approx \lambda\bar{\mu}u_m$ and

$$\tau(1 - u_mu_m^* - u_m^*u_m) = \Big(1 - 2\frac{n-1}{n^2}\Big)^m$$

Now repeating this construction, going further and further out in $\mathcal{H} \simeq l^2(\mathbb{N})$ to find f, g's and using $\frac{1}{2\pi i}\log(\lambda\bar{\mu}) = 2^{-1}, 2^{-2}, 2^{-3}, \dots, 2^{-m}$ (with the last m being that of Theorem 1), one finds a set of matrix units e_{ij}, $i, j = 1, \dots, 2^m$, which may be taken to approximately commute with any finite set in M_{n^∞}, such that $\tau\Big(\sum\limits_{i=1}^{2^m} e_{ii}\Big) > 1 - \varepsilon$ and

$$\sigma(e_{ij}) \approx e_{i+1,j+1}$$

where the addition of indices is modulo 2^m (see [BEK] for details). Putting $e_i = e_{ii}$, Theorem 1 follows.

The proof of Corollary 2 from Theorem 1 now follows the lines of the proofs of [Voi, Lemma 3.3] or [BKRS, Lemma 2.1]. By [Voi, Lemma 3.1], for $D \in \mathcal{F}(M_{n^\infty})$ and $m \in \mathbb{N}$ there are $B_j \in \mathcal{F}(M_{n^\infty})$, $j = 0, 1, \dots, m$ with

$$\begin{aligned}
&B_0 = B_m\,, \\
&d(\sigma(B_i), B_{i+1}) < \frac{5\pi}{m}\,, \qquad i = 0, \dots, m-1\,, \\
&D \subseteq B_i\,, \qquad i = 0, 1, \dots, m\,.
\end{aligned}$$

If $e_0, \ldots, e_m$ is a Rohlin tower approximately commuting with $B_0, \ldots, B_m$ we may assume that $e = \sum_{k=0}^{m-1} e_k$ commute with D by a small perturbation of e, and putting

$$E \approx \sum_{i=0}^{m-1} B_i e_i \,,$$

E has the desired properties.

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